Hitting with probability one for stochastic heat equations with additive noise

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Collaboration with Robert C. Dalang

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Hitting probabilities

Kakutani (1945)

Let $\{\mathbf{B}(t) : t \geq 0\}$ be a d-dimensional Brownian motion. For any Borel set $A \subset \mathbb{R}^d$,

$$\mathrm{P}\{\exists t > 0 : \mathbf{B}(t) \in A\} > 0 \quad \Longleftrightarrow \quad \mathrm{Cap}_{d-2}(A) > 0.$$

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- $\mathrm{Cap}_\beta(A)$ denotes the capacity of A:

$$\mathrm{Cap}_\beta(A) = \left[\inf_{\mu \in \mathcal{P}(A)} \mathcal{E}_\beta(\mu) \right]^{-1},$$

  where

$$\mathcal{E}_\beta(\mu) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} k_\beta(x-y)\mu(dx)\mu(dy), \quad k_\beta(x) = \begin{cases} \|x\|^{-\beta} & \text{if } \beta > 0, \\ \ln(\frac{1}{\|x\|}) & \text{if } \beta = 0, \\ 1 & \text{if } \beta < 0. \end{cases}$$

- If $A = \{z\}$, then

$$\mathrm{Cap}_\beta(\{z\}) = \begin{cases} 1 & \text{if } \beta < 0, \\ 0 & \text{if } \beta \geq 0. \end{cases}$$

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Khoshnevisan-Shi (1999)

Let $\{\mathbf{W}(t) : t \in \mathbb{R}^2_+\}$ be an $\mathbb{R}^d$-valued Brownian sheet. $\forall\, M > 0$, there exists $c = c(M) > 0$ such that for all Borel sets $A \subseteq [-M, M]^d$,

$$c^{-1}\,\mathrm{Cap}_{d-4}(A) \leq \mathrm{P}\{\exists t \in [1,2]^2 : \mathbf{W}(t) \in A\} \leq c\,\mathrm{Cap}_{d-4}(A).$$

Hitting probability for stochastic heat equations

$$\begin{cases} \partial_t \mathbf{u}(t\,,x) = \frac{1}{2}\partial_x^2 \mathbf{u}(t\,,x) + \mathbf{b}(\mathbf{u}(t\,,x)) + \dot{\mathbf{W}}(t\,,x) & t > 0,\ x \in (0,1) \\ \mathbf{u}(t\,,0) = \mathbf{u}(t\,,1) = \mathbf{0}, \\ \mathbf{u}(0,\cdot) = \mathbf{u}_0 \in \mathrm{C}([0,1],\mathbb{R}^d), \end{cases}$$

where $\mathbf{b}(u_1,\ldots,u_d) := (b_1(u_1),\ldots,b_d(u_d))$ with b_i's being bounded and Lipschitz continuous.

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Dalang-Khoshnevisan-Nualart (2007)

Assume $\mathbf{u}_0 \equiv 0$. Fix $M > 0$. There exists $c = c(M) > 0$ such that for all Borel sets $A \subset [-M,M]^d$,

$$\mathrm{P}\{\exists (t,x) \in [1,2] \times [1/4,1/3] : \mathbf{u}(t\,,x) \in A\} \geq c\,\mathrm{Cap}_{d-6}(A).$$

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Theorem 1. (Dalang-P. (2022+)) If $\mathrm{Cap}_{d-6}(A) > 0$, then

$$\mathrm{P}\{\exists (t,x) \in (0,\infty) \times [0,1] : \mathbf{u}(t\,,x) \in A\} = 1.$$

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Proposition. Assume $\|\mathbf{u}_0\| \leq R$. Fix $M > 0$. There exists $c = c(M, R) > 0$ such that for all Borel sets $A \subset [-M, M]^d$,

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- $\{\mathbf{u}(t,\cdot) : t \geq 0\}$ is a Markov process taking values in $\mathrm{C}([0,1],\mathbb{R}^d)$ with invariant measure

$$\mu(\mathrm{d}\phi) = \prod_{i=1}^{d} \exp\left\{2\int_0^1 \tilde{b}_i(\phi_i(x))\mathrm{d}x\right\}\mu_0(\mathrm{d}\phi_i), \quad \phi = (\phi_1,\ldots,\phi_d) \in \mathrm{C}([0,1],\mathbb{R}^d),$$

where μ_0 is the law of Brownian bridge on $[0,1]$ and $\tilde{b}_i$ is the primitive function of b_i ($\mathbf{b} = (b_1,\ldots,b_d)$).

See Da Prato-Zabczyk (1996).

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Lemma. $\mu(\mathrm{B}(0,R)) > 0$ and $\mu(\mathrm{B}(0,R)^c) > 0$ for all $R > 0$.

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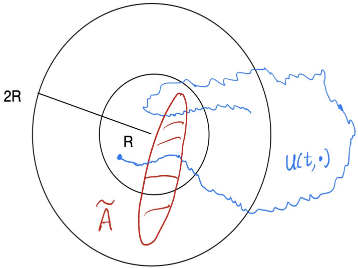
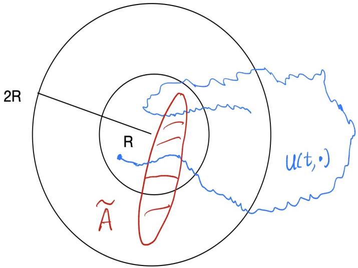
Lemma. $\mu(\mathrm{B}(0,R)) > 0$ and $\mu(\mathrm{B}(0,R)^c) > 0$ for all $R > 0$.

Proposition. $\{\mathbf{u}(t,\cdot) : t \geq 0\}$ is recurrent with respect to $\mathrm{B}(0,R)$ and $\mathrm{B}(0,R)^c$ for all $R > 0$.



$$\tilde{A} := \{\phi \in \mathrm{C}([0,1], \mathbb{R}^d) : \exists\, x \in [0,1] \ \text{ s.t. } \ \phi(x) \in A\}$$

$$\begin{aligned} &\mathrm{P}\{\exists (t,x) \in (0,\infty) \times [0,1] : \mathbf{u}(t\,,x) \in A\} \\ &\quad = \mathrm{P}\{\exists t > 0 : \mathbf{u}(t,\cdot) \in \tilde{A}\} \end{aligned}$$



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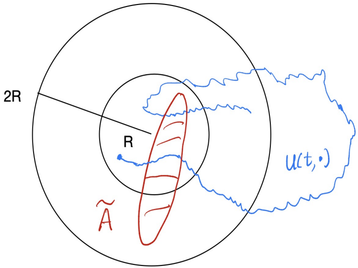
- Let $S_0 = 0$ and for $k \geq 1$,

$$T_k = \inf\{t \geq S_{k-1} : \|\mathbf{u}(t,\cdot)\| > 2R\}, \quad S_k = \inf\{t \geq T_k : \|\mathbf{u}(t,\cdot)\| < R\}.$$

  Then $T_k = S_{k-1} + T_1 \circ \theta_{S_{k-1}}$.
- Set $H_k = \{\exists\, t \in (S_{k-1}, T_k] : \mathbf{u}(t,\cdot) \in \tilde{A}\}$.



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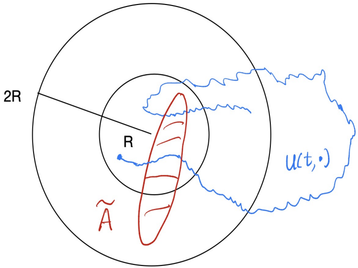
- Set $H_k = \{\exists\, t \in (S_{k-1}, T_k] : \mathbf{u}(t,\cdot) \in \tilde{A}\}$.
- For any $n \geq 1$,

$$\mathrm{P}^{\mathbf{u}_0}\{\exists t > 0 : \mathbf{u}(t,\cdot) \in \tilde{A}\} \geq \mathrm{P}^{\mathbf{u}_0}\left(\bigcup_{k=1}^{n} H_k\right) = 1 - \mathrm{P}^{\mathbf{u}_0}\left(\bigcap_{k=1}^{n} H_k^c\right)$$



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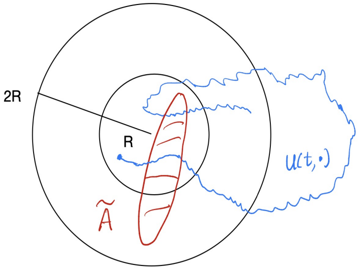
$$= 1 - \mathrm{E}^{\mathbf{u}_0}\left[\left(\prod_{k=1}^{n-1} \mathbf{1}_{H_k^c}\right) \mathrm{P}^{\mathbf{u}_0}\{H_n^c | \mathscr{F}_{S_{n-1}}\}\right]$$



$$\tilde{A} := \{\phi \in C([0,1], \mathbb{R}^d) : \exists x \in [0,1] \text{ s.t. } \phi(x) \in A\}$$

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- Suppose $P^{\mathbf{u}_0}\{H_n | \mathscr{F}_{S_{n-1}}\} \geq c > 0$ for all $n \geq 1$. Then

$$P^{\mathbf{u}_0}\{\exists t > 0 : \mathbf{u}(t, \cdot) \in \tilde{A}\} \geq 1 - (1-c)E^{\mathbf{u}_0}\left[\prod_{k=1}^{n-1} \mathbf{1}_{H_k^c}\right] \geq 1 - (1-c)^n.$$

Letting $n \to \infty$, we obtain $P^{\mathbf{u}_0}\{\exists t > 0 : \mathbf{u}(t, \cdot) \in \tilde{A}\} = 1$.

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- By the strong Markov property,

$$\begin{aligned} P^{\mathbf{u}_0}\{H_n|\mathscr{F}_{S_{n-1}}\} &= P^{\mathbf{u}_0}\{\exists t \in (S_{n-1}, T_n] :\ \mathbf{u}(t,\cdot) \in \tilde{A}|\mathscr{F}_{S_{n-1}}\} \\ &= P^{\mathbf{u}(S_{n-1},\cdot)}\{\exists t \in (0, T_1] :\ \mathbf{u}(t,\cdot) \in \tilde{A}\}. \end{aligned}$$

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- For any $\mathbf{u}_0$ with $\|\mathbf{u}_0\| \leq R$,

$$\begin{aligned} &P^{\mathbf{u}_0}\{\exists t \in (0, T_1] :\ \mathbf{u}(t,\cdot) \in \tilde{A}\} \\ &\geq P^{\mathbf{u}_0}\left\{\{\exists t \in (0, T_1] :\ \mathbf{u}(t,\cdot) \in \tilde{A}\} \bigcap \{\sup_{0\leq t\leq 2} \|\mathbf{u}(t,\cdot)\| \leq 2R\}\right\} \end{aligned}$$

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$$\begin{aligned}\mathrm{P}^{\mathbf{u}_0}\{H_n|\mathscr{F}_{S_{n-1}}\} &= \mathrm{P}^{\mathbf{u}_0}\{\exists t \in (S_{n-1}, T_n] :\ \mathbf{u}(t,\cdot) \in \tilde{A}|\mathscr{F}_{S_{n-1}}\} \\ &= \mathrm{P}^{\mathbf{u}(S_{n-1},\cdot)}\{\exists t \in (0, T_1] :\ \mathbf{u}(t,\cdot) \in \tilde{A}\}.\end{aligned}$$

- For any $\mathbf{u}_0$ with $\|\mathbf{u}_0\| \leq R$,

$$\begin{aligned}&\mathrm{P}^{\mathbf{u}_0}\{\exists t \in (0, T_1] :\ \mathbf{u}(t,\cdot) \in \tilde{A}\} \\ &\geq \mathrm{P}^{\mathbf{u}_0}\left\{\{\exists t \in (0, T_1] :\ \mathbf{u}(t,\cdot) \in \tilde{A}\} \bigcap \{\sup_{0\leq t\leq 2} \|\mathbf{u}(t,\cdot)\| \leq 2R\}\right\} \\ &\geq \mathrm{P}^{\mathbf{u}_0}\left\{\{\exists t \in (0, 2] :\ \mathbf{u}(t,\cdot) \in \tilde{A}\} \bigcap \{\sup_{0\leq t\leq 2} \|\mathbf{u}(t,\cdot)\| \leq 2R\}\right\} \\ &\geq \mathrm{P}^{\mathbf{u}_0}\{\exists (t,x) \in [1,2] \times [1/4, 1/3] :\ \mathbf{u}(t,x) \in A\} - \mathrm{P}^{\mathbf{u}_0}\left\{\sup_{0\leq t\leq 2} \|\mathbf{u}(t,\cdot)\| \geq 2R\right\} \geq c > 0.\end{aligned}$$

Multiplicative noise

$$\begin{cases} \partial_t \mathbf{u}(t\,,x) = \frac{1}{2}\partial_x^2 \mathbf{u}(t\,,x) + \mathbf{b}(\mathbf{u}(t\,,x)) + \boldsymbol{\sigma}(\mathbf{u}(t\,,x))\dot{\mathbf{W}}(t\,,x) & t > 0,\ x \in (0,1) \\ \mathbf{u}(t\,,0) = \mathbf{u}(t\,,1) = \mathbf{0}, \\ \mathbf{u}(0,\cdot) = \mathbf{u}_0 \in \mathrm{C}([0,1],\mathbb{R}^d), \end{cases}$$

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Dalang-Khoshnevisan-Nualart (2009)

Assume $\mathbf{u}_0 \equiv 0$ and conditions [C] on $\mathbf{b}, \boldsymbol{\sigma}$. Fix $M > 0$ and $\epsilon > 0$. There exists $c = c(M, \epsilon) > 0$ such that for all Borel sets $A \subset [-M, M]^d$,

$$\mathrm{P}\{\exists(t,x) \in [1,2] \times [1/4, 1/3] : \mathbf{u}(t\,,x) \in A\} \geq c\,\mathrm{Cap}_{d-6+\epsilon}(A).$$

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Dalang-P. (2021)

Assume $\mathbf{u}_0 \equiv 0$ and conditions [C] on $\mathbf{b}, \boldsymbol{\sigma}$. Fix $M > 0$. There exists $c = c(M) > 0$ such that for all Borel sets $A \subset [-M, M]^d$,

$$\mathrm{P}\{\exists (t,x) \in [1,2] \times [1/4, 1/3] : \mathbf{u}(t\,,x) \in A\} \geq c\,\mathrm{Cap}_{d-6}(A).$$

Multiplicative noise

$$\begin{cases} \partial_t \mathbf{u}(t\,,x) = \frac{1}{2}\partial_x^2 \mathbf{u}(t\,,x) + \mathbf{b}(\mathbf{u}(t\,,x)) + \boldsymbol{\sigma}(\mathbf{u}(t\,,x))\dot{\mathbf{W}}(t\,,x) \quad t > 0,\ x \in (0,1) \\ \mathbf{u}(t\,,0) = \mathbf{u}(t\,,1) = \mathbf{0}, \\ \mathbf{u}(0,\cdot) = \mathbf{u}_0 \in \mathrm{C}([0,1],\mathbb{R}^d), \end{cases}$$

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- $\{\mathbf{u}(t,\cdot) : t \geq 0\}$ has strong Markov property; see Da Prato-Zabczyk (2014).
- Existence and uniqueness of invariance measure: Cerrai (2001), Stannat (2011).
- The recurrence property is not available.